Unit 7

Integrals

SECTION – A

Questions 1 to 10 carry 1 mark each.

1. The value of $\int \frac{1}{1-\sin x}dx$ is

(a) tan x – sec x + C (b) tan x + sec x + C
(c) sec x – tan x + C (d) 1 + sin x + C

Ans. (b) tan x + sec x + C

$$\int \frac{1}{1-\sin x}dx = \int \frac{1+\sin x}{\cos^2 x}dx = \int\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx = \int(\sec^2 x+\sec x\tan x)dx$$

= tan x + sec x + C

2. Evaluate: $\int \frac{1}{\sin^2 x\cos^2 x}dx$

(a) $\sin^2x - \cos^2x + C$ (b) –1
(c) tan x + cot x + C (d) tan x – cot x + C

Ans. (d) tan x – cot x + C

as $\int \frac{1}{\sin^2 x\cos^2 x}dx = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx = \int(\sec^2 x+\text{cosec}^2 x)dx = \tan x - \cot x + C$

3. Given $\int 2^x dx = f(x) + C$, then f (x) is

(a) 2^x (b) $2^x \log_e 2$ (c) $\frac{2^x}{\log_e 2}$ (d) $\frac{2^{x+1}}{x+1}$

Ans. (c) $\frac{2^x}{\log_e 2}$

as $\frac{d}{dx}\left(\frac{2^x}{\log_e 2}\right) = \frac{1}{\log_e 2}\cdot 2^x\cdot\log_e 2 = 2^x$.

4. The value of $\int_8^{13}\frac{\sqrt{21-x}}{\sqrt{x}+\sqrt{21-x}}dx$ is

(a) $\frac{21}{2}$ (b) 0 (c) $\frac{5}{2}$ (d) none of these

Ans. (c) $\frac{5}{2}$

5. The value of $\int_0^{\pi/2} \frac{1}{1+\tan^3 x} dx$ is

(*a*) 0 (*b*) 1 (*c*) $\frac{\pi}{4}$ (*d*) $\frac{\pi}{2}$

Ans. (*c*) $\frac{\pi}{4}$

6. The value of integral $\int_{-1/2}^{1/2} \cos x.\log\left(\frac{1+x}{1-x}\right) dx$ is

(*a*) 0 (*b*) $\frac{1}{2}$ (*c*) $\frac{3}{2}$ (*d*) none of these

Ans. (*a*) 0, as function is odd function.

7. The value of is $\int_0^{\pi/2} \frac{dx}{1+\sin x}$:

(*a*) 0 (*b*) $\frac{1}{2}$ (*c*) 1 (*d*) $\frac{3}{2}$

Ans. (*c*) 1

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos\left(\frac{\pi}{2}-x\right)} = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\cdot\left[\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{-\frac{1}{2}}\right]_0^{\frac{\pi}{2}} = -\tan\left(\frac{\pi}{4}-\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}-0\right) = 1.$$

8. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to:

(*a*) $\frac{e^x}{1+x^2}+\text{C}$ (*b*) $-\frac{e^x}{1+x^2}+\text{C}$ (*c*) $\frac{e^x}{(1+x^2)^2}+\text{C}$ (*d*) $-\frac{e^x}{(1+x^2)^2}+\text{C}$

Ans: (*a*) $\frac{e^x}{1+x^2}+\text{C}$

$$\int e^x\left(\frac{1+x^2-2x}{(1+x^2)^2}\right)dx = \int e^x\left\{\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right\}dx$$

$$f(x) = \frac{1}{1+x^2}; f'(x) = \frac{-2x}{(1+x^2)^2}$$

$$\int e^x\{f(x)+f'(x)\}dx = e^x\cdot f(x)+C = e^x\cdot\frac{1}{1+x^2}+C$$

In the following questions 9 and 10, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as:

(a) Both Assertion (A) and Reason (R) are true and Reason(R) is the correct explanation of assertion (A).
(b) Both Assertion (A) and Reason (R) are true but Reason(R) is not the correct explanation of assertion (A).
(c) Assertion (A) is true but reason (R) is false.
(d) Assertion (A) is false but reason (R) is true.

9. Assertion(A): $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx = 3$

Reason(R) : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Ans: (a) Both A and R are true and R is the correct explanation of A.

10. Assertion(A): $\frac{d}{dx}\left[\int_0^{x^2} \frac{dt}{t^2+4}\right] = \frac{2x}{x^4+4}$

Reason(R): $\int \frac{dx}{x^2+4} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Ans: (a) Both A and R are true and R is the correct explanation of A.

$$\int_0^{x^2} \frac{dt}{t^2+4} = \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)$$

$$\frac{d}{dx}\left[\int_0^{x^2} \frac{dt}{t^2+4}\right] = \frac{d}{dx}\left[\frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)\right]$$

$$= \frac{1}{2} \times \frac{1}{1+\frac{x^4}{4}} \times \frac{2x}{2} = \frac{x}{2} \times \frac{4}{4+x^4} = \frac{2x}{4+x^4}$$

SECTION – B

Questions 11 to 14 carry 2 marks each.

11. Evaluate: $\int \left(\frac{1-\tan x}{1+\tan x}\right)dx$

Ans:

$$\int \left(\frac{1-\tan x}{1+\tan x}\right)dx = \int \frac{\left(1-\frac{\sin x}{\cos x}\right)}{\left(1+\frac{\sin x}{\cos x}\right)}dx = \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}dx$$

$$= \int \frac{1}{t}\,dt, \text{ where } (\cos x + \sin x) = t \text{ and } (\cos x - \sin x)\,dx = dt$$

$$= \log|t| + C = \log|(\cos x + \sin x)| + C.$$

12. Find the value of $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$.

Ans:

$$\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}} = \int_0^4 \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

$$= \left[\log\left|(x+1)+\sqrt{x^2+2x+3}\right|\right]_0^4$$

$$= \{\log|5+\sqrt{27}| - \log|1+\sqrt{3}|\}.$$

13. Evaluate $\int_0^2 f(x)\,dx$, if $f(x) = \begin{cases} -(x-3), & x<2 \\ (x-3), & x>2 \end{cases}$

Ans:

$\int_0^2 (x-3)dx,\ f(x)=\begin{cases}-(x-3), & x<2\\ (x-3), & x>2\end{cases}$

$\therefore \int_0^2 (x-3)dx = -\int_0^2 (x-3)dx = \left[-\frac{x^2}{2}+3x\right]_0^2 = \frac{-1}{2}\left[x^2\right]_0^2+3\left[x\right]_0^2$

$= \frac{-1}{2}(4-0)+3(2-0) \Rightarrow \frac{-4}{2}+6 = -2+6 = 4$

14. Evaluate: $\int \sqrt{\frac{1+x}{1-x}}dx$.

$\int \sqrt{\frac{1+x}{1-x}}dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}dx$

$= \int \frac{1+x}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}}dx$

$= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2}\int \frac{1}{\sqrt{t}}dt$, where $(1-x^2)=t$

$= \sin^{-1}x - \sqrt{t} + C$

$= \sin^{-1}x - \sqrt{1-x^2} + C.$

SECTION – C

Questions 15 to 17 carry 3 marks each.

15. Evaluate: $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

Ans:

Let $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ ... (i)

$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+[\cos(\pi-x)]^2}dx \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right] \Rightarrow I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$ ...(ii)

Adding (i) and (ii), we get $2I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

Also, $x=0 \Rightarrow t=1$ and $x=\pi \Rightarrow t=-1$

$\therefore\ 2I = \int_1^{-1} \frac{-\pi dt}{1+t^2} \Rightarrow I = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$

$\therefore\ I = \frac{\pi}{2}[\tan^{-1}t]_{-1}^{1} = \frac{\pi}{2}[\tan^{-1}(1)-\tan^{-1}(-1)] = \frac{\pi}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$

16. Evaluate: $\int \frac{\sqrt{1-\sin x}}{(1+\cos x)}e^{-x/2}dx$

Ans:

Putting $\frac{-x}{2}=t$, we get $x=-2t$ and $dx=-2dt$.

$$\therefore \quad I=\int \frac{\sqrt{1-\sin x}}{(1+\cos x)} e^{-x/2} dx$$

$$=\int \frac{\sqrt{1-\sin(-2t)}}{\{1+\cos(-2t)\}} e^{t}(-2dt)=-2\int \frac{\sqrt{1+\sin 2t}}{(1+\cos 2t)} e^{t} dt$$

$$=-2\int \frac{\sqrt{\cos^2 t+\sin^2 t+2\sin t\cos t}}{2\cos^2 t} e^{t} dt$$

$$=-2\int \frac{(\cos t+\sin t)}{2\cos^2 t} e^{t} dt=-\int(\sec t+\sec t\tan t)\, e^{t} dt$$

$$=-\int e^{t}\{f(t)+f'(t)\}\, dt, \text{ where } f(t)=\sec t$$

$$=-e^{t} f(t)+C=-e^{-x/2}\sec\left(\frac{-x}{2}\right)+C=-e^{-x/2}\sec\frac{x}{2}+C.$$

17. Evaluate: $\int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx$

Ans:

Putting $\sin x=t$ and $\cos x\, dx=dt$, we get

$$I=\frac{\cos x}{(1-\sin x)(2-\sin x)} dx=\int \frac{dt}{(1-t)(2-t)}.$$

Let $\frac{1}{(1-t)(2-t)}=\frac{A}{(1-t)}+\frac{B}{(2-t)}$

$\Rightarrow\ 1 \equiv A(2-t)+B(1-t).$ ... (i)

Putting $t=1$ in (i), we get $A=1$.

Putting $t=2$ in (i), we get $B=-1$.

$$\therefore \quad \frac{1}{(1-t)(2-t)}=\frac{1}{(1-t)}-\frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx=\int \frac{dt}{(1-t)(2-t)}$$

$$=\int\left\{\frac{1}{(1-t)}-\frac{1}{(2-t)}\right\} dt=\int \frac{dt}{(1-t)}-\int \frac{dt}{(2-t)}$$

$$=-\log|1-t|+\log|2-t|+C$$

$$=\log\left|\frac{2-t}{1-t}\right|+C=\log\left|\frac{2-\sin x}{1-\sin x}\right|+C.$$

SECTION – D

Questions 18 carry 5 marks.

18. Evaluate: $\int_{0}^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx$

Ans:

$$I = \int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)dx = \int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)dx$$

Let $\sin x = t \Rightarrow \cos x dx = dt$

when $x = 0, t = 0$

$x = \frac{\pi}{2}, t = 1$

$$\therefore I = \int_0^1 2t \tan^{-1} t\, dt = 2\left[\tan^{-1} t \int t\, dt - \int\left[\frac{d}{dx}\tan^{-1} t\right]\int t\, dt\right].dt$$

$$= 2\left[\frac{t^2}{2}\tan^{-1} t - \int \frac{t^2}{2(1+t^2)}dt\right]_0^1 = \left[t^2 \tan^{-1} t\right]_0^1 - \int_0^1 \frac{t^2}{(1+t^2)}dt$$

$$= \tan^{-1}1 - \left[\int_0^1 1dt - \int_0^1 \frac{1}{1+t^2}dt\right] = \frac{\pi}{4} - \left[t - \tan^{-1} t\right]_0^1 = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

SECTION – E (Case Study Based Questions)

Questions 19 to 20 carry 4 marks each.

19. Case-Study 1: Read the following passage and answer the questions given below.

A general election of Lok Sabha is a gigantic exercise. About 911 million people were eligible to vote and voter turnout was about 67%, the highest ever.

Let f(x) be the set of all citizens of India who were eligible to exercise their voting right in the general election held in 2019. A relation 'R' is defined on I as follows:

If f(x) is a continuous function defined on [a, b] $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ on the basis of the above information answer the following equations:

(a) Evaluate: $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}dx$ [2]

(b) Find the value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}dx$, a > 0. [2]

Ans: (a)

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x} dx$ ...(i)

Applying property, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-\frac{\pi}{2}-x\right)}{1+e^{\left(\frac{\pi}{2}-\frac{\pi}{2}-x\right)}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(-x)}{1+e^{-x}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \cos x}{1+e^x} dx \quad ...(ii)$$

on adding equation (i) and (ii), we get

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+e^x)\cos x}{1+e^x} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$$

Now, cosx is an even function

$$\therefore \quad 2I = \int_0^{\frac{\pi}{2}} \cos x \, dx \Rightarrow 2I = 2[\sin x]_0^{\frac{\pi}{2}} = 2(1-0) = 2 \Rightarrow I = 1$$

(b)

Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$ ...(i)

Using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(-\pi+\pi-x)}{1+a^{(-\pi+\pi-x)}} dx = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \quad ...(ii)$$

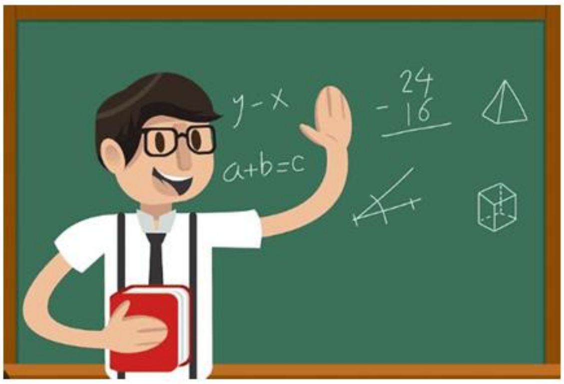
on adding (i) and (ii), we get

$$2I = \int_{-\pi}^{\pi} \cos^2 x \, dx = \left[\frac{1}{2}(x+\sin x \cos x)\right]_{-\pi}^{\pi} = \frac{1}{2}(\pi+0+\pi-0) = \pi$$

$$\Rightarrow I = \frac{\pi}{2}$$

20. Case-Study 2:

Mr. Kumar is a Maths teacher. One day he taught students that the Integral $I = \int f(x)\,dx$ can be transformed into another form by changing the independent variable x to t by substituting



Consider $I = \int f(x)\,dx$

Put $x = g(t)$ so that $\frac{dx}{dt} = g'(t)$ then we write $dx = g'(t)\,dt$

Thus, $I = \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution.

Based on the above information, answer the following questions:

(i) Evaluate: $\int \frac{e^{\tan^{-1} x}}{1+x^2}\,dx$ (2)

(ii) Evaluate: $\int \frac{x^3}{(x^2+1)^3}\,dx$ (2)

OR

(ii) Evaluate: $\int \frac{dx}{x\sqrt{x^6-1}}\,dx$ (2)

Ans: (i)

Put $\tan^{-1} x = t$ so that $\frac{1}{(1+x^2)}\,dx = dt$.

$$\therefore \int \frac{e^{\tan^{-1}x}}{(1+x^2)}\,dx = \int e^t\,dt = e^t + C = e^{\tan^{-1}x} + C.$$

(ii)

Put $(x^2+1) = t$ so that $x^2 = (t-1)$ and $x\,dx = \frac{1}{2}\,dt$.

$$\therefore \int \frac{x^3}{(x^2+1)^3}\,dx = \int \frac{x^2 \cdot x}{(x^2+1)^3}\,dx$$

$$= \frac{1}{2}\int \frac{(t-1)}{t^3}\,dt = \frac{1}{2}\int \frac{1}{t^2}\,dt - \frac{1}{2}\int \frac{1}{t^3}\,dt$$

$$= \frac{-1}{2t} + \frac{1}{4t^2} + C = \frac{-1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C = \frac{-(1+2x^2)}{4(x^2+1)^2} + C.$$

OR

(ii)

Put $x^3 = t$ so that $3x^2dx = dt$ or $x^2dx = \frac{1}{3}\,dt$.

$$\therefore \int \frac{dx}{x \cdot \sqrt{x^6-1}} = \int \frac{x^2}{x^3 \cdot \sqrt{x^6-1}}\,dx$$

[multiplying numerator and denominator by x^2]

$$= \frac{1}{3}\int \frac{1}{t\sqrt{t^2-1}}\,dt = \frac{1}{3}\sec^{-1} t + C = \frac{1}{3}\sec^{-1} x^3 + C.$$